M.Math. IInd year
First semestral Backpaper exam 2020-21
Number Theory
Instructor : B.Sury
Answer ALL SIX questions.

Q 1. Let $p > 2$ be a prime and suppose a is a primitive root mod p. Prove that either a or $a+p$ is a primitive root mod p^n for all $n > 1$.

OR

Prove that a Fermat number $2^{2^n}+1$ can never be a prime power p^r with $r \geq 2$.

Q 2.
(i) Spell out when the Jacobi symbol $(\frac{a}{N})$ is 1.
(ii) Let $p_1, p_2, \cdots, p_n$ be distinct, odd primes. Show the existence of a positive integer N such that the Jacobi symbol $(\frac{N}{p_1 p_2 \cdots p_n}) = -1$.

OR

Let a, b be coprime integers and c be a positive integer. Use the Chinese remainder theorem to prove there exists an integer n such that $a+nb$ is coprime to c.

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Q 3. Let $d \equiv 3 \bmod 4$ be square-free, positive integer. Let $\mathbf{O}$ be the ring $\mathbf{Z}[\frac{1+\sqrt{-d}}{2}]$. Determine the group of units of $\mathbf{O}$.

OR

For a positive integer $n > 2$, show that the number $\Psi(n)$ of positive integers $a \leq n$ such that $(a(a+1), n) = 1$ equals $n \prod_{p|n}(1 - 2/p)$ where the product is over the primes dividing n.

Q 4. Use the quadratic reciprocity law to deduce that if p is any odd prime and $q \equiv 3 \bmod 4$ is a prime, then q is a quadratic residue mod p if and only if $p \equiv \pm a^2 \bmod 4q$ for some odd a relatively prime to q.

OR

Let $p = a^2 + b^2$ be a prime $\equiv 1 \bmod 4$. Prove that $\left(\frac{a+b}{p}\right) = (-1)^{((a+b)^2-1)/8}$.

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Q 5. If $ax^2 + bxy + cy^2$ is a reduced, positive-definite integral form and $au^2 + buv + cv^2 \leq a + |b| + c$ for some $(u, v) = 1$, prove that $au^2 + buv + cv^2$ must be one of $a, c, a \pm |b| + c$.

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Q 6.

(i) Obtain the value of the periodic, simple, continued fraction $[1; \overline{2, 3}]$.

)ii) Obtain the simple, continued fraction expansion of $\sqrt{a^2 + 1}$ for any natural number a.